# Gamma & Beta Functions

## Gamma Function

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \, , n > 0$$

## Properties of Gamma Function

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$
$$\Gamma(n+1) = n\Gamma(n)$$
$$\Gamma(n+1) = n!, \Gamma(1) = 1$$
$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin a\pi}, 0 < a < 1$$

## Examples:

**Evaluate** $\int_0^\infty x^4 e^{-x} x^{n-1} dx$

$$\int_0^\infty x^4 e^{-x} x^{n-1} dx = \int_0^\infty x^{5-1} e^{-x} x^{n-1} dx \ = \Gamma(5) = 4! = 24$$

**Proving that $\Gamma(1/2) = \sqrt{\pi}$**

$\Gamma(1/2) = {}_0\int^\infty x^{1/2-1} e^{-x} dx = {}_0\int^\infty x^{-1/2} e^{-x} dx$

Let $y = x^{½}$, $x = y^{2}$, $dx = 2y\,dy$

$\Gamma(1/2) = \lim_{B\to\infty} \int_0^B y^{-1} e^{-y^2} 2y\,dy$

$= 2 \lim_{B\to\infty} \int_0^B e^{-y^2} dy$

$= 2(\sqrt{\pi}/2) = \sqrt{\pi}$

$_0\int^{\infty} x^{1/2} e^{-x} dx = {}_0\int^{\infty} x^{3/2-1} e^{-x} dx = \Gamma(3/2)$

$3/2 = ½ + 1$

$\Gamma(3/2) = \Gamma(½+1) = ½\ \Gamma(½) = ½\sqrt{\pi}$

**Exercise**

Evaluate $_0\int^{\infty} x^{3/2} e^{-x} dx$

**Example(3)**

Evaluate $_0\int^{\infty} x^{3/2} e^{-x} dx$

$_0\int^{\infty} x^{3/2} e^{-x} dx = {}_0\int^{\infty} x^{5/2-1} e^{-x} dx = \Gamma(5/2)$

$5/2 = 3/2 + 1$

$\Gamma(5/2) = \Gamma(3/2+1) = 3/2\ \Gamma(3/2) = 3/2 \cdot ½\ \Gamma(½) = 3/2 \cdot ½ \cdot \sqrt{\pi} = ¾\sqrt{\pi}$

**Exercise**

Evaluate $_0\int^{\infty} x^{5/2} e^{-x} dx$

## II. Beta Function

$$\mathrm{B(m,n)} = \int_0^1 x^{m-1}(1-x)^{n-1}dx\,, m > 0\ \&\ n > 0$$

Results:

1. $\mathrm{B(m,n)} = \mathrm{B(n,m)}$
2. $\mathrm{B(m,n)} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

**Results:**

(1) $B(m,n) = \Gamma(m)\ \Gamma(n)\ /\ \Gamma(m+n)$

(2) $B(m,n) = B(n,m)$

(3) $_0\int^{\pi/2} \sin^{2m-1}x\ .\ \cos^{2n-1}x\ dx = \Gamma(m)\ \Gamma(n)\ /\ 2\,\Gamma(m+n)$ ; $m>0$ & $n>0$

(4) $_0\int^{\infty} x^{q-1}\ /\ (1+x)\ .\ dx = \Gamma q)\ \Gamma(1-q) = \Pi / \sin(q\pi)$ ; $0<q<1$

## Examples:

**Example(1)**

Evaluate $_0\int^1 x^4\ (1-x)^3\ dx$

**Solution**

$_0\int^1\ x^4\ (1-x)^3\ dx = x^{5-1}\ (1-x)^{4-1}\ dx$

$= B(5,4) = \Gamma(5)\ \Gamma(4)\ /\ \Gamma(9) = 4!\ .\ 3!\ /\ 8! = 3!/(8.7.6.5) = 1/\ (8.7.5) = 1/280$

**Exercise**

Evaluate $_0\int^1 x^2\ (1-x)^6\ dx$

**Example(2)**

Evaluate $I = {}_0\int^1 \left[ 1 / \sqrt[3]{[x^2 (1 - x)]} \right] dx$

**Solution**

$I = {}_0\int^1 x^{-2/3} (1 - x)^{-1/3} dx = {}_0\int^1 x^{1/3 - 1} (1 - x)^{2/3 - 1} dx$

$= B(1/3,2/3) = \Gamma(1/3)\ \Gamma(2/3) / \Gamma(1)$

$\Gamma(1/3)\ \Gamma(2/3) = \Gamma(1/3)\ \Gamma(1- 1/3) = \pi / \sin(\pi/3) = \pi / ( \sqrt{3}/2) = 2\pi / \sqrt{3}$

**Exercise**

Evaluate $I = {}_0\int^1 \left[ 1 / \sqrt[4]{[x^3 (1 - x)]} \right] dx$

**Example(3)**

Evaluate $I = {}_0\int^1 \sqrt{x} . (1 - x) dx$

**Solution**

$I = {}_0\int^1 x^{1/2} (1 - x) dx = {}_0\int^1 x^{3/2 - 1} (1 - x)^{2 - 1} dx$

$= B(3/2 , 2) = \Gamma(3/2)\ \Gamma(2) / \Gamma(7/2)$

$\boldsymbol{\Gamma(3/2) = \frac{1}{2} \sqrt{\pi}}$

$\Gamma(5/2) = \Gamma(3/2+ 1) = (3/2)\ \Gamma(3/2 ) = (3/2) . \frac{1}{2} \sqrt{\pi} = 3\sqrt{\pi} / 4$

$\boldsymbol{\Gamma(7/2) = \Gamma(5/2+ 1) = (5/2)\ \Gamma(5/2 ) = (5/2) . (3\sqrt{\pi} / 4) = 15 \sqrt{\pi} / 8}$

Thus,

$I = (\frac{1}{2} \sqrt{\pi} ) . 1! / (\mathbf{15} \sqrt{\pi} / \mathbf{8}) = 4/15$

**Exercise**

Evaluate $I = {}_0\int^1 \sqrt{x^5} \, . \, (1 - x) \; dx$

## II. Using Gamma Function to Evaluate Integrals

### Example(1)

Evaluate: $I = {}_0\int^{\infty} x^6 \; e^{-2x} \, dx$

Solution:

Letting $y = 2x$, we get

$I = (1/128) \, {}_0\int^{\infty} y^6 \; e^{-y} \, dy = (1/128) \; \Gamma(7) = (1/128) \, 6! = 45/8$

### Example(2)

Evaluate: $I = {}_0\int^{\infty} \sqrt{x} \; e^{-x^3} \, dx$

Solution:

Letting $y = x^3$, we get

$I = (1/3) \, {}_0\int^{\infty} y^{-1/2} \; e^{-y} \, dy = (1/3) \; \Gamma(1/2) = \sqrt{\pi} / 3$

**Example(3)**

Evaluate: $I = \int_0^{\infty} x^m \; e^{-k x^n} dx$

Solution:

Letting $y = k x^n$ , we get

$I = [ 1 / ( n . k^{(m+1)/n}) ] \int_0^{\infty} y^{[(m+1)/n - 1]} \; e^{-y} dy = [ 1 / ( n . k^{(m+1)/n}) ] \; \Gamma[(m+1)/n ]$

## II. Using Beta Function to Evaluate Integrals

**Formulas**

(1) $\int_0^{1} x^{m-1} \; (1 - x)^{n-1} dx = B(m,n) = \Gamma(m) \; \Gamma(n) \; / \; 2 \; \Gamma(m+ n)$ ; $m > 0$ & $n > 0$

(3) $\int_0^{\pi/2} \sin^{2m-1}x \; . \; \cos^{2n-1}x \; dx = (1/2) \, B(m,n)$ ; $m>0$ & $n>0$

(4) $\int_0^{\infty} x^{q-1} \; / \; (1+x) \; . dx = \Gamma(q) \; \Gamma(1-q) = \Pi / \sin(q\pi)$ ; $0 < q < 1$

## Using Formula (1)

## Example(1)

Evaluate: $I = {}_0\int^2 x^2 / \sqrt{(2 - x)} \; . dx$

Solution:

Letting $x = 2y$, we get

$I = (8/\sqrt{2})\, {}_0\int^1 y^2 (1 - y)^{-1/2} dy = (8/\sqrt{2}) . B(3 , 1/2 ) = 64\sqrt{2} /15$

## Example(2)

Evaluate: $I = {}_0\int^a x^4 \sqrt{(a^2 - x^2)} \; . dx$

Solution:

Letting $x^2 = a^2 y$ , we get

$I = (a^6 / 2)\, {}_0\int^1 y^{3/2} (1 - y)^{1/2} dy = (a^6 / 2) . B(5/2 , 3/2 ) = a^6 /3 2$

## Exercise

Evaluate: $I = {}_0\int^2 x \sqrt{(8 - x^3)} \; . dx$

Hint

Lett $x^3 = 8y$

Answer

$I = (8/3)\, {}_0\int^1 y^{-1/3} (1 - y)^{1/3} . dy = (8/3) B(2/3 , 4/3 ) = 16 \pi / ( 9 \sqrt{3} )$

## Using Formula (3)

**Example(3)**

Evaluate: $I = \int_0^{\infty} dx \; / \; (1+x^4)$

Solution:

Letting $x^4 = y$ , we get

$I = (1/4) \int_0^{\infty} y^{-3/4} \; dy \; / \; (1+y) = (1/4) \, . \, \Gamma(1/4) \, . \, \Gamma(1 - 1/4)$

$= (1/4) \, . \, [\, \pi \; / \; \sin(\tfrac{1}{4} \, . \, \pi) \,] = \pi \sqrt{2} \; / \; 4$

## Using Formula (2)

**Example(4)**

a. Evaluate: $I = \int_0^{\pi/2} \sin^3 \; . \; \cos^2 x \; dx$

b. Evaluate: $I = \int_0^{\pi/2} \sin^4 \; . \; \cos^5 x \; dx$

Solution:

a. Notice that: $2m - 1 = 3 \rightarrow m = 2$ & $2n - 1 = 2 \rightarrow m = 3/2$

$I = (1/2) \; B(2 , 3/2) = 8/15$

b. $I = (1/2) \; B(5/2 , 3) = 8/315$

**Example(5)**

a. Evaluate: $I = \int_0^{\pi/2} \sin^6 \; dx$

b. Evaluate: $I = \int_0^{\pi/2} \cos^6 x \; dx$

Solution:

a. Notice that: $2m - 1 = 6 \rightarrow m = 7/2$ & $2n - 1 = 0 \rightarrow m = 1/2$

$I = (1/2)\ B(7/2, 1/2) = 5\pi/32$

b. $I = (1/2)\ B(1/2, 7/2) = 5\pi/32$

**Example(6)**

a. Evaluate: $I = {}_0\int^{\pi} \cos^4 x\ dx$

b. Evaluate: $I = {}_0\int^{2\pi} \sin^8\ dx$

Solution:

a. $I = {}_0\int^{\pi} \cos^4 x = 2\ {}_0\int^{\pi/2} \cos^4 x = 2\ (1/2)\ B\ (1/2, 5/2) = 3\pi/8$

b. $I = I = {}_0\int^{\pi} \sin^8 x = 4\ {}_0\int^{\pi/2} \sin^8 x = 4\ (1/2)\ B\ (9/2, 1/2) = 35\pi/64$

## Details

## I.

**Example(1)**

Evaluate: $I = {}_0\int^{\infty} x^6\ e^{-2x}\ dx$

$x = y/2$

$x^6 = y^6/64$

$dx = (1/2)dy$

$x^6\ e^{-2x}\ dx = y^6/64\ \ e^{-y}\ .\ (1/2)dy$

**Example(2)**

$I = {}_0\int^{\infty} \sqrt{x}\ \ e^{-x\wedge 3}\ dx \quad x=y^{1/3}$

$\sqrt{x}= y^{1/6}$

$dx=(1/3)y^{-2/3}\ dy$

$\sqrt{x} \quad e^{-x^3} dx = y^{1/6} e^{-y} . (1/3)y^{-2/3} dy$

## Example(3)

Evaluate: $I = {}_0\int^{\infty} x^m \quad e^{-k x^n} dx$

$y = k x^n$

$x = y^{1/n} / k^{1/n}$

$x^m = y^{m/n} / k^{m/n}$

$dx = (1/n) y^{(1/n-1)} / k^{1/n} \quad dy$

$x^m \quad e^{-k x^n} dx = ( y^{m/n} / k^{m/n} ) . e^{-y} . (1/n) y^{(1/n-1)} / k^{1/n} \quad dy$

$m/n + 1/n - 1 = (m+1)/n - 1$

$-m/n - 1/n = - (m+1)/n$

$I = [ 1 / ( n . k^{(m+1)/n}) ] \; {}_0\int^{\infty} y^{[(m+1)/n - 1]} \quad e^{-y} dy$

## II. Example(1)

## Example(1)

$I = {}_0\int^{2} x^2 / \sqrt{(2 - x)} \; . dx$

$x = 2y$

$dx=2dy$

$x^2 = 4 y^2$

$\sqrt{(2 - x)} = \sqrt{(2 - 2y)} = \sqrt{2} \quad \sqrt{(1 - y)}$

$x^2 / \sqrt{(2 - x)} \; . dx = 4 y^2 / \sqrt{2} \quad \sqrt{(1 - y)} \quad 2dy$

y=0 when x=0

y=1 when x=2

## Example(2)

Evaluate: $I = {}_0\int^{a} x^4 \sqrt{(a^2 - x^2)} \; . dx$

$x^2 = a^2 y$ , we get
$x^4 = a^4 y^2$
$x= a y^{1/2}$
$dx= (1/2)a y^{-1/2} dy$
$\sqrt{(a^2 – x^2)} = \sqrt{(a^2 – a^2 y)} = a (1 – y )^{1/2}$

$x^4 \sqrt{(a^2 – x^2)} . dx = a^4 y^2 a (1 – y )^{1/2} (1/2)a y^{-1/2} dy$

y=0 when x=0
y=1 when x=a

## Example(3)

$I = {}_0\int^{\infty} dx / ( 1+x^4 )$
$x^4 = y$

$x=y^{1/4}$
$dy= (1/4) y^{-3/4} dy$
$dx / ( 1+x^4 ) = (1 / 4) y^{-3/4} dy / (1 + y )$

## Proofs of formulas (2) & (3)

**Formula (2)**

We have,

$B(m,n) = {}_0\int^1 x^{m-1} (1 – x )^{n-1} dx$

Let $x = \sin^2 y$
Then $dy = 2 \sin x \cos x dx$

&
$x^{m-1} (1 – x )^{n-1} dx = (\sin^2 y)^{m-1} ( \cos^2 y )^{n-1} ( dy / 2 \sin x \cos x )$

$= 2 \sin^{2m-1} y . \cos^{2n-1} y dy$

When x=0 , we have $y = 0$
When x=1, we hae $y = \pi/2$

Thus,

$$I = 2 \; {}_0\int^{\pi/2} \sin^{2m-1} y \, . \, \cos^{2n-1} y \; dy$$

$$I = {}_0\int^{\pi/2} \sin^{2m-1} y \, . \, \cos^{2n-1} y \; dy = B(m,n) / 2$$

**Formula (3)**

We have,

$$I = {}_0\int^{\infty} x^{q-1} \; / \; (1+x) \; dx$$

Let
$y = x / (1+x)$

Hence, $x = y / 1-y$

, $1 + x = 1 + (y / 1-y) = 1/(1-y)$

& $dx = - [ (1- y) – y(-1)] \; / \; (1-y)^2 \; . dy= 1 / (1-y)^2 \; . dy$

whn $x = 0$ , we have $y = 0$
when $x \to \infty$ , we have $y = \lim_{x\to\infty} x / (1+x) = 1$

Thus,

$$I = {}_0\int^{\infty} [ x^{q-1} \; / \; (1+x) ] \, dx = {}_0\int^{\infty} [ ( y / 1-y )^{q-1} \; / \; (1/(1-y)) ] \; . \, 1 / (1-y)^2 \; . dy$$

$$= {}_0\int^{1} [ y^{q-1} \; / (1-y)^{-q} \; ] \; dy$$

$$= B(q , 1-q) = \Gamma(q) \; \Gamma(1-q)$$

## Proving that $\Gamma(1/2) = \sqrt{\pi}$

$$\Gamma(1/2) = {}_0\int^{\infty} x^{1/2-1} \; e^{-x} \, dx = {}_0\int^{\infty} x^{-1/2} \, e^{-x} \, dx$$

Let y $= x^{½,}$ x $= y^{2,}$ dx = 2y dy

$\Gamma(1/2) = \lim_{B\to\infty} \int_0^B y^{-1}\ e - y\hat{}2\ 2y\ dy$

$= 2 \lim_{B\to\infty} \int_0^B e - y\hat{}2\ dy$

$=2\ (\sqrt{\pi}\ / 2\ ) = \sqrt{\pi}$